Def. Cauchy sequence (x_n):

$$\forall_{\varepsilon>0}\ \exists_{\tilde{n}}\ \forall_{m,n>\tilde{n}}\ d(x_n, x_m) < \varepsilon$$

Def. Metric space (X,d) is complete if every Cauchy sequence $(x_n) \subset X$ has a limit in X.

Examples: $(0,1)$ - not complete: $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ has a limit $0 \notin (0,1)$.

$\mathbb{R}^n$ - complete

l_p, l_∞ - complete

$C[0,1] = \{ f: [0,1] \to \mathbb{R},\ f\text{ - continuous} \}$ - complete

Fact. (X,d) complete, $A \subset X$ closed $\Rightarrow$ A - complete

Fact. $\left.\begin{array}{l} A \subset X \text{ compact}, \\ (X,d)\text{ - metric space} \end{array}\right\} \Rightarrow A$ - complete

Proof

A - sequentially compact $\Rightarrow$ every $(x_n) \subset A$ has a convergent subsequence. Then every Cauchy sequence has a convergent subsequence as well. Call it $(x_{n_k}) \xrightarrow[k\to\infty]{} \bar{x}$. Then

$$\forall_{\varepsilon>0}\ \exists_{\tilde{n}}\ \forall_{n, n_k>\tilde{n}}\ d(x_n, x_{n_k}) < \frac{\varepsilon}{2} \text{ and } d(x_{n_k}, \bar{x}) < \frac{\varepsilon}{2}.$$

Hence

$$d(x_n, \bar{x}) \leq d(x_n, x_{n_k}) + d(x_{n_k}, \bar{x}) < \varepsilon \quad \square$$

Def. Banach space = complete normed vector space.

Theorem (Tychonoff). The product of any collection of compact topological spaces is compact wrt the product topology.

Ex. X_i - compact for all i

$$\Downarrow$$

$$X = X_i \times X_j \quad \text{- compact}$$

$$X = \bigtimes_{i=1}^{\infty} X_i \quad \text{- compact}$$

$X = \bigtimes_{i \in I} X_i$ - compact, where I can be any (even uncountable) set of indicators!

(Notation: $\bigtimes_{i \in I}$ - generalized Cartesian product, as in

$$\bigtimes_{i=1}^{n} X_i = X_1 \times X_2 \times \dots \times X_n.$$)

Note: product topology is the coarsest topology (i.e., the topology with fewest open sets) for which all canonical projections $p_i : X \to X_i$ are continuous.

- In a finite-dimensional spaces, such as $\mathbb{R}^n$, the product topology follows from the product metric:

for $z = (x, y)$ and $z' = (x', y')$,

$$d(z, z') = \sqrt{d_1^2(x, x') + d_2^2(y, y')}.$$

- Unfortunately, one cannot generalize the product metric to infinite dimensional spaces. The "box topology" following from $d(x, x') = \sqrt{\sum_{i \in I} d_i^2(x_i, x_i')}$ is finer than the product topology (has more open sets).

"Finite-dimensional Tychonoff theorem":

Let (X, d_1) and (Y, d_2) be two metric spaces.
Define $(X \times Y, d_\pi)$ such that $d_\pi(z, z') = \sqrt{d_1^2(x,x') + d_2^2(y,y')}$, the product metric.
Show that X, Y - compact $\Leftrightarrow$ $X \times Y$ - compact.

(Note. The result is straightforward to generalize to n dimensions. Just prove it by mathematical induction, applying the proof below to $(X_1 \times \ldots \times X_n) \times X_{n+1}$.)

Proof. $\Rightarrow$: X, Y - compact $\Rightarrow$ every sequence $\{x_n\}$, $\{y_n\}$ has a convergent subsequence.

Then take $\{x_{n_k}\} \to \bar{x}$ and specifically take a subsequence $\{y_{n_{k_\ell}}\}$ of the ~~ ~~ subsequence $\{y_{n_k}\}$ such that $\{y_{n_{k_\ell}}\} \to \bar{y}$.

Therefore $\{(x_{n_{k_\ell}}, y_{n_{k_\ell}})\} \subset X \times Y$ is a convergent subsequence of the arbitrary sequence $\{(x_n, y_n)\} \subset X \times Y$.

As $d(x_{n_{k_\ell}}, \bar{x}) \to 0$ and $d(y_{n_{k_\ell}}, \bar{y}) \to 0$, it follows that $d_\pi((x_{n_{k_\ell}}, y_{n_{k_\ell}}), (\bar{x}, \bar{y})) \to 0$.

$\Leftarrow$: $X \times Y$ - compact $\Rightarrow$ every sequence $\{(z_n)\} = \{(x_n, y_n)\} \subset X \times Y$ has a convergent subsequence $\{z_{n_k}\} \to \bar{z}$, where $\bar{z} = (\bar{x}, \bar{y})$. We have $d_\pi(z_{n_k}, \bar{z}) = \sqrt{d_1^2(x_{n_k}, \bar{x}) + d^2(y_{n_k}, \bar{y})}$
As $d_\pi(z_{n_k}, \bar{z}) \to 0$, it follows that

$$d_1(x_{n_k}, \bar{x}) \to 0 \quad \text{and} \quad d_2(y_{n_k}, \bar{y}) \to 0.$$

As (x_n) and (y_n) were arbitrary sequences, we conclude that X and Y are compact. $\square$